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# The absorption of radiation by ballistic electrons in conducting discs

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**Abstract.** The absorption of electromagnetic radiation by small conducting discs is calculated for the case of ballistic electron motion, using a semiclassical analysis valid at frequencies small compared to the plasma frequency $\omega_p$. For a smooth walled disc there are complicated resonance structures, and the absorption coefficient has a low-frequency cut-off at a critical frequency $\omega_c$, which corresponds to the frequency of a circumferential classical orbit of an electron at the Fermi surface. For frequencies satisfying $\omega_c \ll \omega \ll \omega_p$, the absorption coefficient is proportional to $\omega$ if fluctuations due to resonances are averaged over. We also consider a rough-walled disc and some more general shapes: the proportionality to $\omega$ rather than the expected $\omega^2$ is a general feature.

## 1. Introduction

The interaction between electromagnetic radiation and small metallic particles has been subject to intensive investigation, both theoretically and experimentally. At low frequencies, the dominant effect is absorption rather than scattering. Most theoretical analyses of this problem are based upon the Mie theory [1] for the interaction of an electromagnetic wave and a dielectric sphere; a metal particle is regarded as having an imaginary dielectric constant proportional to its conductivity. The application of the Mie theory is only justified if the spherical particle can be regarded as a homogeneous system described by a bulk dielectric constant, and its applicability is questionable if the motion of the charge carriers is ballistic, i.e. if the bulk mean free path of the charge carriers exceeds the size of the particle. A large part of our motivation for this research was to advance our understanding of how to analyse dissipative phenomena in systems with ballistic electron motion.

The case of ballistic motion of the charge carriers has usually been treated by replacing the bulk conductivity with an effective conductivity derived from the Drude formula (discussed clearly by Ashcroft and Mermin [2]). The relaxation time $\tau_r$ for scattering from impurities is replaced by a bounce time $\tau_b \sim a/v_F$, where $a$ is the radius and $v_F$ the Fermi velocity, which is the typical interval between collisions of the charge carrier with the boundary of the particle. This approach was first introduced by Kawabata and Kubo [3], who showed (by comparison with a more precise calculation) that it provides qualitatively correct results for absorption above the plasma frequency. In a recent paper [4], we examined the applicability of this effective conductivity *ansatz* at frequencies below the plasma frequency, where screening of the applied electric field is significant. We found

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that the physics of this situation is complex, with the results depending on the details of the shape of the boundary of the particle and on an additional material-dependent parameter which does not appear in the effective conductivity approximation. Because of the delicate dependence on the shape and composition of the small metal particles predicted by our theory, we felt that it would be interesting to examine this problem for a two-dimensional system (metal discs). Microfabrication procedures could be used to produce very uniform and well characterized samples, whereas spherical particles of well controlled size might be very difficult to prepare. This is not a trivial extension of our earlier work, because in contrast to the three dimensional case the screening charge is not confined to the boundary of the disc. A very surprising consequence of this difference is that for frequencies $\omega$ which are small compared to the plasma frequency $\omega_p$, but large compared to the bounce frequency $\omega_b$, the absorption coefficient $\gamma$ is proportional to $\omega$. This is in contrast to the $\gamma \sim \omega^2$ dependence predicted by the effective conductivity approximation. If the boundary of the disc is smooth enough to allow specular reflection of the electrons, there are also complicated resonance structures superimposed on the $\gamma \sim \omega$ relationship, which we analyse in detail.

Our approach can be summarized as follows. We model the metallic disc as a gas of independent fermions with charge $e$ and isotropic effective mass $m$, confined to a plane and trapped inside a circle of radius $a$ by a confining potential, which is infinite outside the circle and zero inside it. An independent-particle approximation is used, which is valid at high electron densities (which ensures that the Fermi energy is large compared to the Coulomb interaction) and at low temperatures (which ensures that scattering interactions are suppressed by the lack of empty states below the Fermi energy). Each electron is regarded as moving in a self-consistent effective potential which includes the effect of the time-dependent externally applied electric field. Because the potential is time dependent, the energy of the electrons is not a constant of the motion, and we will compute the mean squared change in the energy of the individual electrons resulting from the time dependent perturbation. There is a relationship (discussed in section 2) between this quantity and the increase of the total energy of the electron gas (which is proportional to the absorption coefficient).

Our calculation uses two types of semiclassical approximation, both of which are justified if the radius $a$ is large compared to the Fermi wavelength. Firstly, we use the Thomas–Fermi method to calculate the effective potential. Secondly, we use a classical, rather than a quantum mechanical, method to calculate the change in the energies of the electrons induced by the time-dependent perturbation. Both of these semiclassical approximations are discussed in section 2.

After having discussed the model, in section 3 we present a detailed calculation of the absorption coefficient for discs with a smooth boundary. In section 4 we discuss some generalizations, indicating the universality of the $\gamma \sim \omega$ relationship, and we present an analysis of absorption by a rough-walled disc. Section 5 briefly discusses the experimental observability of the results.

## 2. The semiclassical approach

We assume that although the conducting disc is small enough for the electron motion to be ballistic, it is large enough for a semiclassical analysis to be applicable, in which information about the the classical dynamics of the electrons is used instead of quantum states.

The electrons are assumed to behave as a set of independent particles whose dynamics is determined by a single-particle effective Hamiltonian of the form

$$H = p^2/2m + V_0(r) + V_1(r)\sin\omega t \tag{2.1}$$

where the time-dependent term represents the effect of the externally applied electric field of magnitude $\mathcal{E}(t) = \mathcal{E}_0 \sin\omega t$, which we will assume to be polarized along the $x$ axis, in the plane of the disc. The potential $V_1(r)$ experienced by an electron due to the applied field is not simply $e\mathcal{E}_0 x$ because of the screening effect of the other electrons, which we will consider in due course. One approach to analysing the effect of the time-dependent perturbation would be to calculate the eigenfunctions and eigenvalues of (2.1) at $t = 0$, and to apply time dependent perturbation theory; this would be a difficult calculation, because of the necessity to calculate matrix elements of the perturbation. Under suitable conditions it is possible to simplify the calculation by making use of the correspondence principle. This is justified when two conditions are satisfied: firstly, the scale size of the fluctuations of the potential energy should be large compared with the de Broglie wavelength, and secondly the typical spacing of the energy levels should be sufficiently small that they can be regarded as a quasi-continuum. Both of these conditions are satisfied for a sufficiently large disc.

The change $\Delta E(t)$ in the energy of a single electron at time $t$ will therefore be calculated classically:

$$\Delta E(t) = \int_0^t \mathrm{d}t' \, \frac{\partial H}{\partial t'} = \omega \int_0^t \mathrm{d}t' \; V_1(r(t'))\cos\omega t' \tag{2.2}$$

where $r(t)$ is the classical trajectory of the electron.

In [4] we obtained a semiclassical formula for the total energy absorbed $\Delta E_T(t)$ by the electron gas due to the action of the perturbation:

$$\Delta E_T(t) = h^{-2}\Omega(E_F)\langle\Delta E^2(E_F)\rangle \tag{2.3}$$

where $\langle\Delta E^2(E_F)\rangle$ is the second moment of the changes in energy $\Delta E(t)$ experienced by individual electrons in the neighbourhood of the Fermi energy, and $\Omega(E)$ is the weight of the energy shell at energy $E$:

$$\Omega(E) = \int \mathrm{d}r \int \mathrm{d}p \; \delta(E - H_0(r, p)). \tag{2.4}$$

The quantity $h^{-2}\Omega(E_F)$ is the density of states $n(E)$ per unit area per spin at the Fermi energy multiplied by the area of the particle. For a disc of radius $a$, this is

$$h^{-2}\Omega(E_F) = \pi a^2 n(E_F) = ma^2/2\hbar^2. \tag{2.5}$$

The calculations performed in [4] and here show that $\Delta E_T$ depends linearly on time, apart from unimportant periodic fluctuations, so the average value of $\mathrm{d}E_T/\mathrm{d}t$ is a constant. The absorption coefficient $\gamma(\omega)$ is clearly proportional to the time-averaged energy absorption $\langle\mathrm{d}E_T/\mathrm{d}t\rangle$ in a single particle. We envisage that the most probable application of these results would be to a layer of discs on a surface. Because there is no standardized definition of $\gamma(\omega)$ for this geometry, we will only quote results for the rate of absorption of energy by a single disc.

Before applying equations (2.2)–(2.5) we must develop a theory for the effective potential $V_1(r)$ due to the externally applied electric field. For this we will use the Thomas–Fermi approximation [2], in which it is assumed that the effective potential $V_1(r)$ (assumed small) varies slowly on the length scale of the Fermi wavelength (the validity of this assumption will be discussed in more detail later). The small perturbation $V_1(r)$ of the effective potential causes a change in the density of electrons $\delta n(r)$

$$\delta n(r) = \left(\frac{\partial N}{\partial E}\right)\Big|_{E_F} V_1(r) \tag{2.6}$$

where $N(E_F)$ is the integrated density of electrons for a free electron gas with Fermi energy $E_F$, including spin:

$$N(E_F) = mE_F/\pi\hbar^2. \tag{2.7}$$

The change $\delta n(r)$ in the density of the electron gas results in a corresponding change $\delta q = e\,\delta n$ in the charge density. If the disc is sufficiently large, we can assume that the change $\delta q(r)$ in the charge density is that which would be predicted by classical electrostatics. Because we are only interested in frequencies $\omega$ which are small compared to the plasma frequency, we can assume that $\delta q$ is the same as for a conducting disc in a static electric field: this is [5]

$$\delta q(r,\theta) = \frac{4\epsilon_0 \mathcal{E} r\cos\theta}{\pi\sqrt{a^2-r^2}} \tag{2.8}$$

where $(r,\theta)$ are plane polar coordinates and the electric field is taken to be along the $x$ direction. Note that, in contrast to the conducting sphere, the charge density extends inside the disc rather than being confined to a thin layer at the boundary. This justifies the use of the Thomas–Fermi approximation, everywhere except at the edge. Combining (2.5)–(2.7), we obtain

$$V_1(r) = \frac{4\epsilon_0\mathcal{E}\hbar^2}{me}\frac{r\cos\theta}{\sqrt{a^2-r^2}}. \tag{2.9}$$

This potential has a divergence at the edge of the disc. The simple Thomas–Fermi approach would break down here, because the potential is not slowly varying, but this approximation is expected to be adequate throughout the interior of the disc. The divergence is integrable, in the sense that the energy $\Delta E(t)$, computed from (2.2), will remain finite (except for the special case of the trajectory around the circumference of the disc). The corrections to (2.9) near $r = a$ need not therefore be considered. This completes our description of the semiclassical approximations involved.

## 3. Calculation of the rate of absorption

To calculate the rate of absorption of energy we use (2.2) and (2.9) to calculate the second moment of the change in the single-particle energies, and then apply (2.3).

For the unperturbed classical motion the total energy $E$ and the angular momentum $J$ are constants of the motion. Figure 1 illustrates a typical trajectory. The polar angle $\theta$ is incremented by an amount $2\phi$ between bounces. The angle $\phi$ is related to $J$ by

$$J = mav_F\cos\phi \tag{3.1}$$

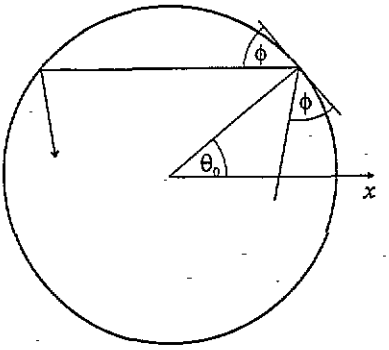


**Figure 1.** Illustration of a classical trajectory for a particle bouncing inside a smooth disc and the definition of the angles $\phi$ and $\theta_0$.

and the corresponding time between bounces $\tau$ is

$$\tau = (2a/v_F)\sin\phi. \tag{3.2}$$

Simple geometry shows that the radial coordinate $r$ is a periodic function of time with period $\tau$:

$$r(t) = a\sqrt{1 + \frac{v_F^2 t^2}{a^2} - \frac{2v_F t\sin\phi}{a}} \tag{3.3}$$

where $t$ is measured from the most recent bounce. The polar angle $\theta$ has the time dependence

$$\begin{aligned}\sin(\theta-\theta_0) &= \frac{(v_F t/a)\cos\phi}{\sqrt{1+(v_F t/a)^2-(2v_F t/a)\sin\phi}}\\ \cos(\theta-\theta_0) &= \frac{1-(v_F t/a)\sin\phi}{\sqrt{1+(v_F t/a)^2-(2v_F t/a)\sin\phi}}\end{aligned} \tag{3.4}$$

where $\theta_0$ is the polar angle of the previous bounce. It is useful to separate the motion in the $\theta$ direction into a secular component and a periodic component:

$$\theta'(t) = \theta(t) - \theta_0 - 2\phi t/\tau \tag{3.5}$$

where $\theta'(t)$ is periodic with period $\tau$, and $\theta_0$ is the polar angle of the initial bounce. Using (2.2) and (2.9), an expression for $\Delta E$ can be obtained:

$$\Delta E(t) = \frac{4\hbar^2\omega\epsilon_0\mathcal{E}_0}{me}\int_0^t \mathrm{d}t' \frac{r(t')\cos\omega t'\cos\theta(t')}{\sqrt{a^2-r^2(t')}}. \tag{3.6}$$

This expression is not directly tractable analytically but can be expanded as a Fourier series. Substituting for $\theta$ from (3.5) gives

$$\begin{aligned}\Delta E(t) = C\Bigg[&\int_0^t \mathrm{d}t' \frac{r(t')\cos\theta'(t')\cos\omega t'\cos(\theta_0+2\phi t'/\tau)}{\sqrt{a^2-r^2(t')}}\\ &-\int_0^t \mathrm{d}t' \frac{r(t')\sin\theta'(t')\cos\omega t'\sin(\theta_0+2\phi t'/\tau)}{\sqrt{a^2-r^2(t')}}\Bigg]\end{aligned} \tag{3.7}$$

with $C = 4\hbar^2\omega\epsilon_0\mathcal{E}_0/me$. The parts of the integrands which are periodic with period $\tau$ can now be written as Fourier series:

$$\begin{aligned}\frac{r(t)\cos\theta'(t)}{\sqrt{a^2-r^2(t)}} &= \sum_{n=-\infty}^{\infty} a_n(\phi)\exp(2\pi i n t/\tau)\\ \frac{r(t)\sin\theta'(t)}{\sqrt{a^2-r^2(t)}} &= \sum_{n=-\infty}^{\infty} b_n(\phi)\exp(2\pi i n t/\tau).\end{aligned} \tag{3.8}$$

Using the expressions (3.3) and (3.4) allows $a_n$ and $b_n$ to be obtained as

$$\begin{aligned}a_n(\phi) &= \frac{1}{2\sin\phi}\int_0^1 \mathrm{d}T\left[\sqrt{\frac{1-T}{T}}\cos(2\phi T)+\sqrt{\frac{T}{1-T}}\cos[2\phi(1-T)]\right]\mathrm{e}^{-2\pi i n T}\\ &= \frac{1}{\sin\phi}\int_0^1 \mathrm{d}T\sqrt{\frac{1-T}{T}}\cos(2\phi T)\cos(2\pi n T) = a_{-n}(\phi)\\ b_n(\phi) &= \frac{1}{2\sin\phi}\int_0^1 \mathrm{d}T\left[-\sqrt{\frac{1-T}{T}}\sin(2\phi T)+\sqrt{\frac{T}{1-T}}\sin[2\phi(1-T)]\right]\mathrm{e}^{-2\pi i n T}\\ &= \frac{\mathrm{i}}{\sin\phi}\int_0^1 \mathrm{d}T\sqrt{\frac{1-T}{T}}\sin(2\phi T)\sin(2\pi n T) = -b_{-n}(\phi)\end{aligned} \tag{3.9}$$

where $T$ is the scaled variable $t/\tau$. These coefficients can be obtained numerically and can also be formally expressed in terms of confluent hypergeometric functions [6, 7]. After some algebra, the integral (3.7) can be obtained as

$$\begin{aligned}\Delta E = \frac{C\tau}{4}\sum_{n=-\infty}^{\infty}\Bigg\{ & c_n(\phi)\mathrm{e}^{\mathrm{i}\theta_0}\left[\frac{\exp\mathrm{i}(2\phi t/\tau+\omega t+2\pi n t/\tau)-1}{\mathrm{i}(2\phi+\omega\tau+2\pi n)}\right.\\ &\left.+\frac{\exp\mathrm{i}(2\phi t/\tau-\omega t+2\pi n t/\tau)-1}{\mathrm{i}(2\phi-\omega\tau+2\pi n)}\right]\\ &+ d_n(\phi)\mathrm{e}^{-\mathrm{i}\theta_0}\left[\frac{\exp\mathrm{i}(-2\phi t/\tau+\omega t+2\pi n t/\tau)-1}{\mathrm{i}(-2\phi+\omega\tau+2\pi n)}\right.\\ &\left.+\frac{\exp\mathrm{i}(-2\phi t/\tau-\omega t+2\pi n t/\tau)-1}{\mathrm{i}(-2\phi-\omega\tau+2\pi n)}\right]\Bigg\}\end{aligned} \tag{3.10}$$

where the real coefficients $c_n(\phi)$ and $d_n(\phi)$ are defined by $c_n(\phi) = a_n(\phi)+\mathrm{i}b_n(\phi)$, $d_n(\phi) = a_n(\phi) - \mathrm{i}b_n(\phi)$. By grouping complex conjugate pairs of terms and using the symmetry properties of $a_n$ and $b_n$, (3.10) can be expressed in a more convenient form involving summing over positive values of $n$ only:

$$\begin{aligned}\Delta E = \frac{C\tau c_0(\phi)}{2}\Bigg[ & \frac{\cos\theta_0\sin(\omega t+2\phi t/\tau)+\sin\theta_0(\cos(\omega t+2\phi t/\tau)-1)}{(\omega\tau+2\phi)}\\ &+\frac{\cos\theta_0\sin(-\omega t+2\phi t/\tau)+\sin\theta_0(\cos(-\omega t+2\phi t/\tau)-1)}{(-\omega\tau+2\phi)}\Bigg]\\ &+\frac{C\tau}{2}\sum_{n=1}^{\infty}c_n(\phi)\frac{1}{2\pi n+2\phi\pm\omega\tau}[\cos\theta_0\sin(2\pi n t/\tau+2\phi t/\tau\pm\omega t)\\ &+\sin\theta_0(\cos(2\pi n t/\tau+2\phi t/\tau\pm\omega t)-1)]\\ &+\frac{C\tau}{2}\sum_{n=1}^{\infty}d_n(\phi)\frac{1}{2\pi n-2\phi\pm\omega\tau}[\cos\theta_0\sin(2\pi n t/\tau-2\phi t/\tau\pm\omega t)\\ &-\sin\theta_0(\cos(2\pi n t/\tau-2\phi t/\tau\pm\omega t)-1)]\end{aligned} \tag{3.11}$$

where terms with both signs are included in the sum. Equation (3.11) has resonances at values of $\phi$ for which the denominators vanish. The analysis given below shows that these resonances do not overlap, so that in the neighbourhood of a resonance a valid approximation to $\Delta E$ is obtained by retaining only a single resonant term. The average of $\Delta E^2$ is dominated by contributions from the resonances, and because only one of the terms is large at any one value of $\phi$, this average will be approximated by sum of the squares of the individual terms of (3.11); cross terms will be neglected. It is convenient to average the squared terms over the initial angle $\theta_0$. This gives

$$\begin{aligned}\langle \Delta E^2 \rangle_{\theta_0} \simeq \frac{C^2\tau^2}{2}\Bigg[ & c_0(\phi)^2\left(\frac{\sin^2 \frac{1}{2}(2\phi t/\tau + \omega t)}{(\omega\tau + 2\phi)^2} + \frac{\sin^2 \frac{1}{2}(2\phi t/\tau - \omega t)}{(-\omega\tau + 2\phi)^2}\right) \\ & + \sum_{n=1}^{\infty} c_n^2(\phi)\left(\frac{\sin^2 \frac{1}{2}(2\phi t/\tau + \omega t + 2\pi n t/\tau)}{(\omega\tau + 2\phi + 2\pi n)^2}\right. \\ & \left. + \frac{\sin^2 \frac{1}{2}(2\phi t/\tau - \omega t + 2\pi n t/\tau)}{(-\omega\tau + 2\phi + 2\pi n)^2}\right) \\ & + \sum_{n=1}^{\infty} d_n^2(\phi)\left(\frac{\sin^2 \frac{1}{2}(-2\phi t/\tau + \omega t + 2\pi n t/\tau)}{(\omega\tau - 2\phi + 2\pi n)^2}\right. \\ & \left. + \frac{\sin^2 \frac{1}{2}(-2\phi t/\tau - \omega t + 2\pi n t/\tau)}{(-\omega\tau - 2\phi + 2\pi n)^2}\right)\Bigg] = \tfrac{1}{2}C^2\tau^2 F(\phi) \end{aligned} \tag{3.12}$$

where $\langle\ \rangle_{\theta_0}$ denotes an average over $\theta_0$. The expression (3.12) explicitly shows the resonance properties referred to above. When there is a resonance between the field frequency and the classical motion of the electrons, one of the denominators in this expression becomes small. The resonance conditions are

$$2\phi_k^{+*} + \omega\tau = \pm 2k\pi \qquad 2\phi_k^{-*} - \omega\tau = \pm 2k\pi \qquad k = 0, 1, \ldots . \tag{3.13}$$

Similar resonances also occur in the case of ballistic electrons in a sphere, discussed in [4]. The interpretation of these resonances is discussed in more detail below.

To obtain the value of $\langle \Delta E^2(t) \rangle$ is necessary to perform the phase space average

$$\langle \Delta E^2 \rangle = \frac{\int \mathrm{d}\alpha\ \Delta E^2 \delta(E - E_\mathrm{F})}{\int \mathrm{d}\alpha\ \delta(E - E_\mathrm{F})} \tag{3.14}$$

where the $\alpha$ are the phase space coordinates. A suitable canonical set of coordinates for this calculation is $(E, J, t_0, \theta_0)$, where $t_0$ is the initial time coordinate of the particle, measured from the previous bounce. The relation between $(x, y, p_x, p_y)$ and $(t_0, \theta_0)$ is

$$x - p_x t_0/m = a\cos\theta_0 \qquad y - p_y t_0/m = a\sin\theta_0. \tag{3.15}$$

The average over $\theta_0$ has already been performed in (3.12). The denominator of (3.14) can readily be evaluated by converting the integral over $J$ to an integral over $\phi$, giving

$$\frac{1}{2\pi}\int \mathrm{d}\alpha\ \delta(E - E_\mathrm{F}) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi\ ma v_\mathrm{F} \sin\phi \int_0^\tau \mathrm{d}t_0 = \pi m a^2 \tag{3.16}$$

where the range of $\phi$ is set to account for positive and negative values of $J$. Performing the integral with respect to $t_0$ and substituting for $\tau$ gives the numerator as

$$\frac{4ma^4C^2}{v_\mathrm{F}^2}\int_{-\pi/2}^{\pi/2} \mathrm{d}\phi\ \sin^4\phi\ F(\phi). \tag{3.17}$$

The evaluation of (3.17) can be illustrated by considering the first term,

$$\int_{\pi/2}^{\pi/2} \mathrm{d}\phi \sin^4\phi \frac{\sin^2[\frac{1}{2}(2\phi t/\tau + \omega t)]}{(\omega\tau + 2\phi)^2}. \tag{3.18}$$

By substituting $y = (\phi t/\tau + \omega t/2)$, the integral for this term can be obtained as

$$\frac{v_F t}{8a} \int \mathrm{d}y \frac{c_0^2(\phi)\sin^4\phi}{[\sin\phi - \phi\cos\phi]} \frac{\sin^2 y}{y^2}. \tag{3.19}$$

Taking the slowly varying parts outside the integral allows the integral over $y$ to be performed giving the value $\pi$. Repeating the same procedure for the remaining terms and combining with (3.16) and (3.17) gives

$$\langle \Delta E^2 \rangle = \frac{8a\hbar^4\omega^2\epsilon_0^2\mathcal{E}_0^2 t}{m^2e^2v_F}\Bigg[c_0^2(\phi_0^{+*})f_+(\phi_0^{+*}) + c_0^2(\phi_0^{-*})f_-(\phi_0^{-*}) + \sum_{n=1}^{\infty}(c_n^2(\phi_n^{+*}) + d_n^2(\phi_n^{+*}))f_+(\phi_n^{+*}) + (c_n^2(\phi_n^{-*}) + d_n^2(\phi_n^{-*}))f_-(\phi_n^{-*})\Bigg] \tag{3.20}$$

with

$$f_\pm(\phi^*) = \left|\frac{\sin^4\phi^*}{\sin\phi^* - \cos\phi^*(\phi^* \mp n\pi)}\right| = \left|\frac{\sin^3\phi^*}{1 \pm \omega a\cos\phi^*/v_F}\right| \tag{3.21}$$

where the $\phi_n^{+*}, \phi_n^{-*}$ are the allowed solutions of (3.11); the second equality follows from (3.13) and (3.2). Using (2.3) $\langle \mathrm{d}E_T/\mathrm{d}t\rangle$ can be obtained as

$$\left\langle \frac{\mathrm{d}E_T}{\mathrm{d}t}\right\rangle = \frac{4a^3\hbar^2\omega^2\epsilon_0^2\mathcal{E}_0^2}{me^2v_F}\Bigg[c_0^2(\phi_0^{+*})f_+(\phi_0^{+*}) + c_0^2(\phi_0^{-*})f_-(\phi_0^{-*}) + \sum_{n=1}^{\infty}(c_n^2(\phi_n^{+*}) + d_n^2(\phi_n^{+*}))f_+(\phi_n^{+*}) + (c_n^2(\phi_n^{-*}) + d_n^2(\phi_n^{-*}))f_-(\phi_n^{-*})\Bigg]. \tag{3.22}$$

The sum in (3.22) is taken over all possible solutions of (3.13) at frequency $\omega$.

Figure 2(a) shows numerical results obtained for the lowest six resonance bands of (3.21); both the individual bands and the total absorption coefficient are displayed. Figure 2(b) shows the absorption coefficient obtained from the first 50 bands. Below the frequency $\omega_0 = v_F/a$ there is no absorption. The onset of the lowest band corresponds to synchrotron acceleration of an electron in a circumferential orbit with $\phi = 0$; this band has a $\phi^{-1}$ divergence at the lower edge, corresponding to a $(\omega - \omega_c)^{-1/2}$ divergence in the absorption coefficient. The lowest band has a finite upper cut-off at $\phi = \pi/2$, which corresponds to the electron bouncing along the diameter of the disc. The remaining bands have onset at $\phi = \pi/2$ and no upper cut-off. The divergence of the lowest band is associated with the singularity of the potential (2.9) at the edge of the particle and would be smoothed out in a more realistic model.

At high frequency it can be seen that both the resonance peak heights and the smoothed absorption coefficient are proportional to the frequency. This behaviour can be obtained

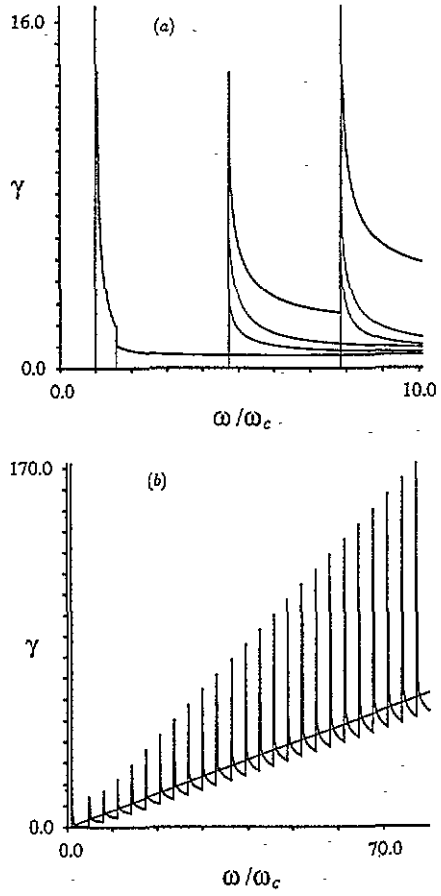


**Figure 2.** Absorption coefficient as a function of frequency: (a) first six bands, with the individual bands included, (b) first 50 bands. The units of absorption coefficient correspond to the pre-multiplier in (3.22) being set to $\omega^2$.

directly from the asymptotic forms of the coefficients $a_n$ and $b_n$ in (3.9). For $a_n$, the change of variable $T' = 2\pi nT$ gives

$$a_n \sin\phi = \frac{1}{\sqrt{2\pi n}} \int_0^{2\pi n} \mathrm{d}T' \frac{\sqrt{1 - T'/2\pi n}}{\sqrt{T'}} \cos\left(\frac{\phi T'}{\pi n}\right) \cos T'$$
$$\sim \frac{1}{\sqrt{2\pi n}} \int_0^{\infty} \mathrm{d}T' \frac{\cos T'}{\sqrt{T'}} = \frac{1}{2\sqrt{n}} \tag{3.23}$$

where the approximate form is the the large-$n$ limit and the numerical factor is obtained from the relation of the integral obtained to a Fresnel integral [7]. A similar argument for $b_n$ shows that $b_n \to 0$ as $n \to \infty$. At large values of the resonance index $n$, $n$ is proportional to $\omega$ so that each new resonance contributing to (3.22) in this limit gives a contribution proportional to $\omega$ as observed. It is possible to use the asymptotic expression obtained above to estimate the average slope of the absorption coefficient versus frequency plot; this quantitity is relevant to experiments which might not resolve the resonance structures. Using (3.23) the asymptotic form of (3.22) is

$$\left\langle \frac{dE_T}{dt} \right\rangle = \tfrac{1}{2} D\omega^2 (I_+ + I_-) \tag{3.24}$$

with $D = 4a^3\hbar^2\epsilon_0^2\mathcal{E}_0^2/me^2v_F$ and

$$I_\pm = \int_0^\infty dn \frac{\sin^2\phi}{n(\sin\phi - \cos\phi(\phi \pm n\pi))}. \tag{3.25}$$

Using the resonance condition $n\pi = (\omega a/v_F)\sin\phi \mp \phi$ gives

$$I_\pm = \int_0^{\pi/2} d\phi \frac{\sin\phi}{\omega a \sin\phi/v_F \mp \phi} \sim \frac{\pi v_F}{2\omega a} \tag{3.26}$$

where the approximation is the large-$\omega$ limit, and hence

$$\left\langle \frac{dE_T}{dt} \right\rangle = \frac{2\pi a^2\hbar^2\epsilon_0^2\mathcal{E}_0^2\omega}{me^2}. \tag{3.27}$$

This average high-frequency absorption is shown in figure 2(b). Using (3.23) it is also possible to derive the result that the onset of each pair of resonances gives a spike of height twice the average value.

## 4. Generalization to other shapes

A surprising feature of the calculation presented above is that the absorption coefficient is proportional to $\omega$ in the high-frequency limit. A naïve argument would suggest that, because the dipole induced by the field is independent of frequency in the low-frequency limit, the current is proportional to $\omega$. If the disc could be characterized by a frequency-independent resistance, this would imply that the energy dissipated is proportional to $\omega^2$. In this section we will look at the explanation for the $\omega$ frequency dependence from a more physical viewpoint. It will be shown that this dependence is a general characteristic of two-dimensional particles, and that it stems from a universal form for the divergence of the charge density at the edge of a particle.

We first consider the form of the charge density near the boundary. The electrostatic potential $v$ is constant on the surface of the particle. We consider a system of local Cartesian coordinates in the neighbourhood of the edge of the particle, such that the conducting region is the plane $y = 0, x > 0$. The potential is independent of the third coordinate in the neighbourhood of the edge, i.e. $v = v(x, y)$, and without loss of generality we can set $v(x, 0) = 0$ for $x \geqslant 0$. Because $v$ is dependent only on $x$ and $y$, we can write $z = x + iy$, and use the fact that if $w = u + iv = f(z)$, where $f(z)$ is an analytic function, then both $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation, $\nabla^2 v = 0$. In this context the correct function is $f(z) = z^{1/2}$, which gives the following relationships between $u$, $v$ and $x$, $y$:

$$x = u^2 - v^2 \qquad y = 2uv \tag{4.1}$$

which can be solved for $v(x, y)$. The potential $v(x, y)$ given in parametric form by (4.1) has the property that the equipotentials are folded around the line $y = 0, x \geqslant 0$, and converge toward this line in the limit $v \to 0$. The potential in the neighbourhood of a straight edge

of a conducting plate will always converge to a multiple of $v(x, y)$. The charge density on the plate is proportional to the discontinuity of $\partial v/\partial y$ at the plate. From (4.1), we find

$$\frac{\partial v}{\partial y} = \frac{y}{2v(x+v^2)} \tag{4.2}$$

and that in the limit $v \to 0$, $v \sim |y|/\sqrt{x}$, so that

$$\left.\frac{\partial v}{\partial y}\right|_{y=0\pm} = \frac{\pm 1}{\sqrt{x}}. \tag{4.3}$$

The charge density in the neighbourhood of the edge of a smooth plate can therefore be written in the form

$$\delta q(r) \sim C(s)/\sqrt{\xi} \tag{4.4}$$

where $s$ measures the distance around the circumference, $\xi$ measures the distance from the nearest point on the boundary, and $C(s)$ is a function which must be obtained from the global solution of the electrostatic problem. We note for later reference that for a disc the function $C(s)$ is

$$C(s) = \frac{4\sqrt{a}\epsilon_0 \mathcal{E}\cos\theta}{\sqrt{2\pi}} \tag{4.5}$$

where $s = a\theta$.

When the frequency $\omega$ is large compared to the characteristic frequency of collisions with the boundary $\omega_c$, we will show that the changes in the energy of the electron occur in the neighbourhood of the collisions with the boundary. In this case the limited information about the charge distribution contained in (4.4) is sufficient to determine the response of the system. Consider the change in the energy of an electron which strikes the boundary at time $t = t_b$, between $t = t_b - \Delta t$ and $t = t_b + \Delta t$, where $\Delta t$ is small enough to ensure that there are no other collisions with the boundary. Assume that the electron strikes the boundary with angle of incidence $\phi$. Using (2.2), (2.6) and (4.4), the change in the energy of the electron is

$$\delta E = \frac{\pi\hbar^2\omega}{me} C(s) \int_{t_b-\Delta t}^{t_b+\Delta t} \mathrm{d}t \frac{1}{\sqrt{\xi(t)}} \cos\omega t. \tag{4.6}$$

The distance from the boundary is $\xi(t) = v_F|t - t_b|\sin\phi$. In the limit $\omega t_0 \gg 1$, the energy transferred is therefore

$$\delta E = \frac{\pi\hbar^2\omega}{me\sqrt{v_F \sin\phi}} C(s) \int_{-\infty}^{\infty} \mathrm{d}t \frac{\cos\omega t}{\sqrt{t - t_b}} = \frac{\pi\hbar^2\omega^{1/2}}{me\sqrt{v_F \sin\phi}} C(s)\ I\ \cos\omega t_b \tag{4.7}$$

where

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{|\sqrt{x}|} \cos x = \sqrt{2\pi}. \tag{4.8}$$

The total energy transferred in time $t$ is the sum of contributions from individual bounces of the form (4.7). Because (4.7) contains the factor $\omega^{1/2}$, the absorption coefficient obtained from (2.3) will always be proportional to $\omega$ when $\omega_c \ll \omega \ll \omega_p$.

We illustrate the application of (4.7) by re-deriving (3.27). From (4.7), the total energy change experienced by an electron with angular momentum $J$ in time $t$ is

$$\Delta E(t) = \sum_j \delta E_j = \frac{4\sqrt{\pi a\omega}\hbar^2\epsilon_0\mathcal{E}_0}{me\sqrt{v_F \sin\phi}} \sum_{j=1}^{N} \cos\omega t_j \cos\theta_j \tag{4.9}$$

where $t_j$ and $\theta_j$ are the times and polar angles of the $N = (t/\tau)$ collisions with the walls. The sum in (4.9) should be evaluated as a geometric series, leading to the same type of sum over resonances as was treated in section 3. We will adopt a simpler procedure, and treat the sum as if the terms were uncorrelated. In appendix B of reference [4], we show that this procedure is justified in the high frequency limit, where the resonances are dense. We therefore write

$$\langle \Delta E^2(t)\rangle_{\theta_0} = \frac{16\pi a\hbar^4\epsilon_0^2\mathcal{E}_0^2\omega}{m^2e^2v_F \sin\phi} N\langle\cos^2\omega t\rangle\langle\cos^2\theta\rangle = \frac{2\pi\hbar^4\epsilon_0^2\mathcal{E}_0^2\omega t}{m^2e^2\sin^2\phi}. \tag{4.10}$$

In order to use (2.3) we require the phase space average of $\Delta E^2(t)$. It is convenient to use the canonical coordinates $E, t_0, J, \theta_0$ discussed in section 3: (4.10) is already averaged over $\theta_0$, and averaging $\langle \sin^{-2}\phi\rangle$ over $t_0$ and $J$ gives

$$\langle \Delta E^2(t)\rangle = 4\pi\hbar^4\epsilon_0^2\mathcal{E}_0^2\omega t/m^2e^2. \tag{4.11}$$

Finally, using (2.5) and (2.3), the rate of absorption is

$$\left\langle \frac{dE_T}{dt} \right\rangle = \frac{2\pi a^2\hbar^2\epsilon_0^2\mathcal{E}_0^2\omega}{me^2} \tag{4.12}$$

which is in agreement with (3.27).

It is also possible to perform a similar calculation for a disc with ergodic motion. For this calculation we assume that the charge density is the same as for a smooth-walled disc but that surface roughness causes angular momentum conservation to break down so that the collision angle $\phi$ is different for each bounce. Repeating the above calculation with this assumption gives

$$\begin{aligned}\langle \Delta E^2(t)\rangle &= \frac{16\pi a\hbar^4\epsilon_0^2\mathcal{E}_0^2\omega}{m^2e^2v_F} \sum_j \cos^2\omega t_j \cos^2\theta_j \frac{1}{|\sin\phi_j|} \\ &= \frac{4\pi a\hbar^4\epsilon_0^2\mathcal{E}_0^2\omega}{m^2e^2v_F} N\left\langle \frac{1}{|\sin\phi|} \right\rangle \end{aligned} \tag{4.13}$$

where the $t_j$ are the bounce times and $N$ is the number of bounces, $tv_F/\langle d\rangle$ with $\langle d\rangle$ the average distance between bounces. This involves averaging $d = 2a\sin\phi$. Using the phase space coordinates $(E, J, t_0, \theta_0)$ as before gives the average over $\phi$ as

$$\langle d\rangle = \frac{2a\int_{-\pi/2}^{\pi/2} d\phi\ |\sin^3\phi|}{\int_{-\pi/2}^{\pi/2} d\phi\ \sin^2\phi} = \frac{16a}{3\pi}. \tag{4.14}$$

Similarly,

$$\left\langle \frac{1}{|\sin\phi|} \right\rangle = \frac{\int_{-\pi/2}^{\pi/2} d\phi\ |\sin\phi|}{\int_{-\pi/2}^{\pi/2} d\phi\ \sin^2\phi} = \frac{4}{\pi}. \tag{4.15}$$

Combining these results gives

$$\langle \Delta E^2(t) \rangle = 3\pi\hbar^4\epsilon_0^2\mathcal{E}_0^2\omega/me^2 \tag{4.16}$$

and

$$\left\langle \frac{\mathrm{d}E_T}{\mathrm{d}t} \right\rangle = \frac{3\pi a^2\hbar^2\epsilon_0^2\mathcal{E}_0^2\omega}{me^2}. \tag{4.17}$$

This result is very similar numerically to that for the average $\langle \mathrm{d}E_T/\mathrm{d}t \rangle$ for a smooth disc; unlike the smooth disc there are no resonances. It may appear surprising that this result does not depend upon the nature of the ergodic motion. This is related to the assumption that the frequency is much higher than the bounce frequency: the rapidly oscillating term $\cos \omega t_j$ in (4.9) makes successive bounces against the boundary appear to be uncorrelated.

## 5. Concluding remarks

Our results indicate that for frequencies $\omega$ satisfying $\omega \gg \omega_c = v_F/a$ and $\omega \ll \omega_p$, the absorption coefficient of a two-dimensional conducting particle with ballistic electrons is proportional to frequency. This conclusion depends on the electron motion being ballistic, but it is not dependent on the shape of the particle or on whether reflections at the boundary are specular. This is a surprising conclusion because it is at variance with the prediction of the Kawabata–Kubo *ansatz*, which predicts an $\omega^2$ dependence.

We also gave a detailed analysis of the semiclassical model for a disc with a smooth, specularly reflecting boundary. In this case we found that the response is determined by resonances between the classical motion and the applied field. There is a low-frequency cut-off at the synchrotron frequency $\omega_c = v_F/a$. The first resonance, at $\omega_c$, corresponds to a synchrotron acceleration of the electrons in a circumferential orbit. This resonance diverges in the simple Thomas–Fermi approximation because of the divergence of the effective potential on the boundary, which would be removed in a more sophisticated theory. The other resonances do not diverge and in the limit $\omega \gg \omega_c$ they overlap and sum to an average $\omega$ dependence.

These resonance effects are dependent on the particles being quite precisely circular in shape, and upon the reflections of electrons at the boundary being specular. This may be experimentally realizable in conducting discs prepared by lithography of semiconductor systems in which a two-dimensional electron gas is confined to a surface layer. Because the density of charge carriers is low in semiconductors, the Fermi energy is small and the Fermi wavelength is correspondingly large. If the Fermi wavelength is large compared with the scale size of the irregularities of the boundary, reflections will be specular. At least one experiment has been done involving transport in a laterally structured two dimensional electron gas which can only be explained convincingly by assuming ballistic motion of independent carriers with specular reflections at the boundaries [8].

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